

Lifetime of flying particles in canonical Doubly Special Relativity

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Abstract

We discuss the corrections to the lifetime of unstable elementary particles in some models of doubly special relativity. We assume that the speed of light is invariant and that the position coordinates transform in such a way to ensure the invariance of the deformed symplectic structure of phase space.

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1 Introduction

Some years ago, the idea that special relativity should be modified in such a way that the Planck energy κ be an invariant parameter like the speed of light was proposed [1]. Models based on this idea were named doubly special relativity (DSR). Originally, the idea was implemented through a deformation of Lorentz transformations acting on momentum space, which in turn implied the deformation of the dispersion relations of elementary particles [1]. In particular, it was assumed that only proper Lorentz transformations were deformed, while rotations were realized canonically. Several different models of this kind were proposed, based on different deformations [2, 3]. The analysis performed in these papers was limited to the momentum sector of phase space, while the spacetime realization of the deformations was not discussed.

This led to some criticism, based on the observation that nonlinear transformations of the momenta could bring back the action of the Lorentz group to its canonical form [4], making therefore the theory trivial. Even accepting this point of view, it should however be noted that realizations of DSR on position space give rise to predictions that are certainly not equivalent to those of special relativity.

Later, in fact, different realizations on spacetime of the DSR idea were proposed. First of all, it was realized that existing models based on κ -Poincaré algebras and κ -Minkowski spacetime [5] fitted very well the DSR axioms. Other classical realizations were mainly based on the definition of appropriate transformation laws for the positions coordinates, which enforced the invariance of the (not necessarily canonical) symplectic structure of phase space¹ [7, 8, 9, 10]. A similar proposal was also advanced in [11]. In this paper, however, the transformation laws for the positions coordinates were not assumed to preserve the symplectic structure, but rather the scalar product of positions with momenta. A critical discussion of these models can be found in [12].

It also appeared that the most natural realization of the DSR idea is in terms of noncommutative spacetime. Recently, a realization of DSR on noncommutative spacetime with no modifications of the dispersion relations has also been proposed [13].

In general, the phenomenological consequences of DSR depend on the

¹This idea was first outlined in [6] in the context of κ -Minkowski spacetime.

spacetime model adopted [14], and especially on the definition of the velocity of a particle [15, 16, 17]. In this letter we consider a specific proposal for the spacetime realization of DSR [9, 8, 18], based exclusively on classical particle mechanics, that starting from any given momentum space realization of DSR, defines its position space realization by requiring covariance of the phase space coordinates. To distinguish this model from the numerous different realizations of DSR, we call it canonical DSR.

This proposal is based on two requests: define the transformation law of position coordinates in such a way that they leave the symplectic form of phase space invariant, and obtain a definition of velocity compatible with its identification as a parameter of Lorentz transformations [15, 16, 17]. The last demand enforces the deformation of the canonical structure of phase space, yielding nonvanishing Poisson brackets between position coordinates [8, 17], that can be interpreted as a classical counterpart of noncommutative geometry. It is also important to notice that our definition of velocity is such that the speed of light is really a constant, thus avoiding the problems related to a variable speed of light present in some spacetime realizations of DSR [11]. In this letter, we shall not expose in detail the formalism of canonical DSR, but refer to the above-cited papers [9, 8, 18].

The result of our analysis is that in the best known examples of DSR [2, 5], canonical DSR implies nontrivial corrections to the time of flight of unstable particles with respect to the predictions of special relativity. This possibility was already put forward in ref. [18]. A discussion of time of flight of particles based on the spacetime realization of DSR of ref. [11] is given in ref. [19]. Due to the considerable differences in the two approaches (in particular the variability of the speed of light assumed in [11]), the results of [19] do not apply here.

In the following we raise and lower indices with the flat metric of signature $(+, -, -, -)$. Greek indices run from 0 to 3 and latin indices from 1 to 3.

2 The Magueijo-Smolín model

As a first example of application of the formalism, we consider the Magueijo-Smolín (MS) model [2], that is the simplest realization of DSR in momentum space. It can be characterized by the deformation of the transformation law of the momentum of a particle under boosts. For a boost in the x^1 direction

with rapidity parameter ξ , the momentum transforms as [2]

$$\begin{aligned} p'_0 &= \frac{p_0 \cosh \xi + p_1 \sinh \xi}{\Delta(p_\mu)}, & p'_1 &= \frac{p_1 \cosh \xi + p_0 \sinh \xi}{\Delta(p_\mu)}, \\ p'_2 &= \frac{p_2}{\Delta(p_\mu)}, & p'_3 &= \frac{p_3}{\Delta(p_\mu)}, \end{aligned} \quad (1)$$

where

$$\Delta(p_\mu) = 1 + \frac{p_0(\cosh \xi - 1) + p_1 \sinh \xi}{\kappa}. \quad (2)$$

The dispersion relation of the MS model, invariant under the transformations (1) is

$$\frac{p_0^2 - p_i^2}{\left(1 - \frac{p_0}{\kappa}\right)^2} = m^2, \quad (3)$$

where m is the Casimir mass. This is related to the rest energy M of the particle by

$$m = \frac{M}{1 - \frac{M}{\kappa}}. \quad (4)$$

From (1) one can derive a relation between the rapidity parameter and the energy p_0 of a particle in a frame related to its rest frame by a boost of parameter ξ [16],

$$p_0 = \frac{M\gamma}{1 + \frac{M}{\kappa}(\gamma - 1)}, \quad (5)$$

where $\gamma \equiv \cosh \xi$. Inverting, one obtains the parameter γ as a function of the energy p_0 ,

$$\gamma = \frac{p_0 \left(1 - \frac{M}{\kappa}\right)}{M \left(1 - \frac{p_0}{\kappa}\right)}. \quad (6)$$

Defining the 3-velocity v_i of a particle in a suitable way, it is possible to identify γ with its classical expression, $\gamma = (1 - v_i^2)^{-1/2}$ [15, 16]. This condition is necessary if one requires that the transformation law of the velocity of a particle under boosts be independent of its mass [16], and permits to identify the velocity with the parameter of the Lorentz transformations [15]. Moreover, it preserves the Einstein formula for the composition of velocities and implies that the speed of light is independent of the energy. Such definition of velocity can be obtained by deforming the canonical symplectic

structure as [7, 8, 17]

$$\begin{aligned}\{x^0, x^i\} &= \frac{x^i}{\kappa}, & \{p_0, p_i\} &= 0, & \{x^0, p_0\} &= 1 - \frac{p_0}{\kappa}, \\ \{x^i, p_j\} &= \delta_j^i, & \{x^0, p_i\} &= -\frac{p_i}{\kappa}, & \{x^i, p_0\} &= 0,\end{aligned}\quad (7)$$

and imposing that the position variables x^μ transform in such a way that, combined with (1), leave invariant the corresponding symplectic structure [8, 9], namely²

$$\begin{aligned}x'^0 &= \Delta(p_\mu)(x^0 \cosh \xi - x^1 \sinh \xi), & x'^1 &= \Delta(p_\mu)(-x^0 \sinh \xi + x^1 \cosh \xi), \\ x'^2 &= \Delta(p_\mu) x^2, & x'^3 &= \Delta(p_\mu) x^3,\end{aligned}\quad (8)$$

with $\Delta(p_\mu)$ given by (2). We observe that, as usual in DSR theories, a consistent definition of the Lorentz transformations of the position coordinates must be momentum dependent.

Choosing the Hamiltonian proportional to the Casimir invariant (3), the Hamilton equations then read [7, 8]

$$\dot{x}_0 = \frac{p_0}{\left(1 - \frac{p_0}{\kappa}\right)^2}, \quad \dot{x}_i = \frac{p_i}{\left(1 - \frac{p_0}{\kappa}\right)^2}, \quad (9)$$

$$\dot{p}_0 = \dot{p}_i = 0, \quad (10)$$

The classical definition of 3-velocity follows,

$$v_i \equiv \frac{\dot{x}_i}{\dot{x}_0} = \frac{p_i}{p_0}. \quad (11)$$

It is also easy to verify that the metric [7, 8]

$$ds^2 = \left(1 - \frac{p_0}{\kappa}\right)^2 d\bar{s}^2, \quad (12)$$

where $d\bar{s}^2$ is the Minkowski metric, is invariant under the deformed transformations (1), (8). A noticeable property of the metric (12) is its momentum dependence. This also is a common feature of DSR models and has led to a

²The same transformations were proposed in [11], although starting from different assumptions.

proposal for their generalization to include gravity [20]. Note that in our formalism the causal structure is not affected by the dependence on momenta. The metric must be interpreted as that experienced by a particle of energy p_0 . A similar situation arises in scalar-tensor gravity, where particles with different scalar coupling experience different metric structures. Using (5) one may also interpret the metric as dependent on the mass and the velocity of the particles, rather than momenta, obtaining a structure similar to, but not coincident with, that of Finsler geometry [21].

Let us now proceed to calculate the proper time of a particle with coordinates x^μ in terms of the time measured by an observer of coordinates x'^μ at rest in the laboratory. For a small displacement dx^μ , using (8) one has, since $dx^i = 0$,

$$dx^0 = \Delta(p_\mu) \gamma dx^0, \quad (13)$$

where $\Delta(p_\mu)$ is given by (2) with $p_\mu = (M, 0, 0, 0)$. Calling t the laboratory time x^0 , and τ the proper time x^0 , one has, more explicitly,

$$dt = \gamma \left[1 + \frac{M}{\kappa}(\gamma - 1) \right] d\tau. \quad (14)$$

The same result can be obtained from the invariance of the line element (12). Equating (12) calculated in the laboratory and rest frames, it results

$$\left(1 - \frac{p_0}{\kappa} \right)^2 (1 - v_i^2) dt^2 = \left(1 - \frac{p_0}{\kappa} \right)^2 \gamma^{-2} dt^2 = \left(1 - \frac{M}{\kappa} \right)^2 d\tau^2. \quad (15)$$

Substituting (5), one recovers (14).

From a phenomenological point of view, it may be more useful to write dt as a function of the energy p_0 of the particle. Using (6), one gets

$$dt = \frac{p_0}{M} \frac{\left(1 - \frac{M}{\kappa} \right)^2}{\left(1 - \frac{p_0}{\kappa} \right)^2} d\tau \approx \frac{p_0}{M} \left(1 + 2 \frac{p_0 - M}{\kappa} \right) d\tau. \quad (16)$$

It is then evident that the formalism of canonical DSR implies corrections to the time of flight formula of special relativity of order p_0/κ . In particular, an observer at rest in the laboratory measures a value of the lifetime of an unstable particle greater than that predicted by special relativity. Although the corrections are too small to be detected at present, they are in principle observable.

3 The Lukierski-Nowicki-Ruegg model

An analogous calculation can be done for the Lukierski-Nowicki-Ruegg (LNR) model [5]. This analysis has already been performed in ref. [18], but here we rederive that result starting from the invariance of the line element.

In the LNR model a deformed boost in the x^1 direction acts on the momentum variables as [22]

$$\begin{aligned} p'_0 &= p_0 + \kappa \log \Gamma, & p'_1 &= \frac{p_1 \cosh \xi + \frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} + \frac{p_k^2}{\kappa^2}\right) \sinh \xi}{\Gamma(p_\mu)}, \\ p'_2 &= \frac{p_2}{\Gamma(p_\mu)}, & p'_3 &= \frac{p_3}{\Gamma(p_\mu)}, \end{aligned} \quad (17)$$

where

$$\Gamma(p_\mu) = \frac{1}{2} \left(1 + e^{-2p_0/\kappa} - \frac{p_k^2}{\kappa^2}\right) + \frac{1}{2} \left(1 - e^{-2p_0/\kappa} + \frac{p_k^2}{\kappa^2}\right) \cosh \xi + \frac{p_1}{\kappa} \sinh \xi. \quad (18)$$

The dispersion relation invariant under the deformed boosts reads

$$4\kappa^2 \sinh^2 \frac{p_0}{2\kappa} - p_k^2 e^{p_0/\kappa} = m^2, \quad (19)$$

with m the Casimir mass, related to the rest mass M by

$$m = 2\kappa \sinh \frac{M}{2\kappa}. \quad (20)$$

The definition of velocity consistent with its identification as the parameter of the Lorentz transformations is [15, 17, 23]

$$v_i = \frac{2p_i e^{p_0/\kappa}}{\kappa \left(e^{p_0/\kappa} - \cosh \frac{M}{\kappa}\right)}, \quad (21)$$

and the parameter $\gamma = \cosh \xi$ of the transformation between the laboratory and the rest frames of a particle of energy p_0 is given by [16]

$$\gamma = \frac{e^{p_0/\kappa} - \cosh \frac{M}{\kappa}}{\sinh \frac{M}{\kappa}}. \quad (22)$$

The definition (21) can be obtained by postulating the Poisson brackets [18]

$$\begin{aligned}\{x_0, x_i\} &= 2 \frac{p_i x_0}{\kappa^2} - \left(1 + \frac{p_k^2}{\kappa^2}\right) \frac{x_i}{\kappa}, \quad \{p_0, p_i\} = 0, \\ \{x_0, p_0\} &= \frac{1}{2} \left(1 + e^{-2p_0/\kappa} + \frac{p_k^2}{\kappa^2}\right), \quad \{x_0, p_i\} = -\frac{p_i}{\kappa} e^{-2p_0/\kappa}, \\ \{x_i, p_0\} &= \frac{p_i}{\kappa}, \quad \{x_i, p_j\} = -e^{-2p_0/\kappa} \delta_{ij}.\end{aligned}\tag{23}$$

Note that that these Poisson brackets are different from the standard ones adopted in κ -Minkowski space [5].

Given the symplectic structure (23), the transformation of the position coordinates under a boost in the x^1 direction, contravariant with respect to (17) are [18]

$$\begin{aligned}x'^0 &= \frac{x^0 \cosh \xi - x^1 \sinh \xi}{\Gamma(p_\mu)}, \quad x'^1 = \frac{x^1 \cosh \xi - x^0 \sinh \xi}{\Gamma(p_\mu)}, \\ x'^2 &= \frac{x^2}{\Gamma(p_\mu)}, \quad x'^3 = \frac{x^3}{\Gamma(p_\mu)},\end{aligned}\tag{24}$$

Choosing the Hamiltonian proportional to the Casimir invariant (19), the Hamilton equations read

$$\begin{aligned}\dot{x}_0 &= \frac{\kappa e^{2p_0/\kappa}}{8m} \left(1 + e^{-2p_0} - \frac{p_k^2}{\kappa^2}\right)^2 \left(1 - e^{-2p_0/\kappa} + \frac{p_k^2}{\kappa^2}\right), \\ \dot{x}_i &= \frac{e^{2p_0/\kappa}}{4m} \left(1 + e^{-2p_0} - \frac{p_k^2}{\kappa^2}\right)^2 p_i,\end{aligned}\tag{25}$$

from which one can recover the velocity (21). The metric invariant under (17) and (24) reads [18]

$$ds^2 = \frac{16 e^{-2p_0/\kappa}}{(1 + e^{-2p_0} - p_1^2/\kappa^2)^4} d\bar{s}^2.\tag{26}$$

For a particle of rest mass M the metric can be written, using (19) and (20),

$$ds^2 = \frac{e^{2p_0/\kappa}}{(2 \cosh(M/\kappa) - 1)^2} d\bar{s}^2.\tag{27}$$

Equating (27) evaluated in the laboratory frame and in the rest frame, one gets, using the same notations as in previous section,

$$\frac{e^{p_0/\kappa}}{2 \cosh(M/\kappa) - 1} \frac{dt}{\gamma} = \frac{e^{M/\kappa}}{2 \cosh(M/\kappa) - 1} d\tau. \quad (28)$$

On the other hand, the energy p_0 of the particle measured in the laboratory frame can be written in terms of γ as (cfr. (22))

$$e^{p_0/\kappa} = \cosh(M/\kappa) + \gamma \sinh(M/\kappa). \quad (29)$$

Substituting in (28) one gets

$$dt = \frac{2\gamma}{1 + e^{-2M/\kappa} + \gamma(1 - e^{-2M/\kappa})} d\tau. \quad (30)$$

This is the result obtained in [18] by a different method. Using (22) one can also write this relation in terms of the energy p_0 as

$$dt = 2 \frac{1 - \cosh \frac{M}{\kappa} e^{-p_0/\kappa}}{1 - e^{-2M/\kappa}} d\tau \sim \frac{p_0}{M} \left(1 - \frac{p_0 - M}{\kappa} \right) d\tau. \quad (31)$$

Notice that in this case the correction has opposite sign with respect to that obtained for the MS model, and hence the predicted lifetimes of unstable flying particles are smaller than the ones of special relativity.

4 Conclusions

We have shown that the formalism of canonical DSR induces nontrivial corrections to the time of flight of an unstable particle, with respect to the predictions of special relativity. This shows the nontriviality of DSR when a spacetime realization is given. The corrections depend on the specific model of DSR under consideration, and are related to the deformation of the Lorentz invariance. It is easy to see in fact that models of DSR that preserve the Lorentz invariance, like the Snyder model [24] do not exhibit such effect.

It must also be remarked that the results we have obtained depend strongly on the spacetime realization of DSR, and in particular on the definition of velocity. Different spacetime models will lead to different conclusions, see for example [19]. Unfortunately, there is no agreement yet on this topic

in the literature, and the eventual choice of a specific realization should be considered a matter of experiment.

On the other hand, an experimental check of our results seems out of reach at present. The most favourable situation for measurements of quantum-gravity effects seems to be the detection of neutrinos associated with gamma ray burst, whose energies are around 10^5 GeV [25]. In the atmosphere, they could produce muons or tau particles of similar energies. Even if the lifetime of these particles could be measured, the corrections to the relativistic formula would be of order 10^{-12} , which seems beyond observational power. The effects predicted could however be detectable if some mechanism fixes the scale of κ well below the Planck energy.

Finally, we remark that, although our realization of DSR solves the problems related to the definition of velocity, it leaves other puzzles open, in particular the definition of multiparticle states and the macroscopic limit of the theory. The first problem could become relevant when applying our results to composite particles.

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